

Nineteen Independent Sections from Two Quadratic Covers of an Elliptic K3 Surface

C. R. Hill  *

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Abstract

I give two explicit quadratic covers of a rank-17 elliptic K3 fibration. Each cover adds a section; on their genus-one compositum, the seventeen inherited sections and the two new ones are independent. The parameter curve has a rational point of infinite order. This yields infinitely many elliptic curves over $\mathbb{Q}$ of rank at least 19, and the fibers can be chosen to have trivial rational torsion. The exact section coordinates and a SageMath verification script are available in the companion repository. The numerical bound $C(0) \geq 19$ was already known; this is a different explicit construction.

1 Introduction

This started with [curve #3900 on the ICARM Elliptic Curve Rank Leaderboard](#), submitted by Ritik Jain. I used that curve as a starting point for a sequence of two-neighbor constructions. One of the resulting elliptic fibrations had seventeen independent sections over $\mathbb{Q}(t)$. A search on that fibration then found a fiber at

$$t_0 = \frac{15122}{33609}$$

with a stored lower bound of 27 independent rational points. That fiber was a useful clue: it gave a rational place to look for quadratic multisections. The rank-27 computation is part of the discovery story, not a premise of the rank-19 proof below.

Two multisections worked. Each is a quadratic cover of the t -line and adds one rational section after base change. Their compositum is a genus-one curve. If I parametrize the first quadratic cover, the second one becomes a quartic equation. This is where the quartic enters; the two original covers themselves are quadratic.

Here is the precise result. Let $E/\mathbb{Q}(t)$ be the short Weierstrass model whose coefficient vectors are given in the permanent data archive [3]. That archive also gives seventeen sections $P_1, \dots, P_{17}$ and the two sections constructed below. Let C be the smooth projective model of

$$r_1^2 = q_1(t), \quad r_2^2 = q_2(t), \tag{1}$$

where q_1, q_2 are displayed in Section 2.

Theorem 1. *The curve C has genus one and infinitely many rational points. Over $\mathbb{Q}(C)$, the elliptic curve E has at least nineteen independent sections. Infinitely many rational fibers E_t have rank at least nineteen and trivial rational torsion.*

*Website: <https://hillaxiom.com> | Email: crhillresearch@gmail.com

The proof uses exact identities and heights. The online certificate [3] lists the coefficients and rational functions in full; its verifier checks the defining equations, the height matrices, the quadratic involutions, and the finite-field counts. The long coefficients stay in that companion archive so the argument here can be read without pages of integers.

Elkies previously reached the bound $C(0) \geq 19$ using paired quadratic base changes on a rank-17 K3 surface [2]. My result does not increase that number. The constructions are distinct: the parent height determinants are 1152 here and 948 for Elkies. Both fibrations have only irreducible singular fibers, and Elkies proves geometric Picard number 19 for his surface. If the parent K3 surfaces were isomorphic, the determinant ratio would be a rational square by the Néron–Severi discriminant formula [5]. But $1152/948 = 96/79$ is not a square. Dujella lists Hill (2026) alongside Elkies in his table of infinite families [1].

2 The K3 surface and two quadratic covers

Write the parent fibration as

$$E_t : y^2 = x^3 + A(t)x + B(t), \quad A, B \in \mathbb{Q}[t], \quad (2)$$

with $\deg A = 8$ and $\deg B = 12$. The vectors of coefficients of A and B , in increasing degree order, are the fields `a4_low_to_high` and `a6_low_to_high` in `data/parent-rank17.json` [3]. This is a compact way to specify a model whose coefficients are too long to help a reader on the printed page. The same file gives the rational-function coordinates of the seventeen sections. Its discriminant has degree 24 and is squarefree, so the corresponding elliptic K3 surface has 24 fibers of type I_1 and no reducible fibers.

The two branch polynomials are

$$q_1(t) = 30015061720809t^2 - 20692173881368t + 3538696407184, \quad (3)$$

$$q_2(t) = 46784549960425t^2 - 29557331565600t + 4702049024256. \quad (4)$$

Both are squarefree and $\gcd(q_1, q_2) = 1$. Thus their classes in $\mathbb{Q}(t)^\times / \mathbb{Q}(t)^{\times 2}$ are independent. The compositum

$$\mathbb{Q}(C) = \mathbb{Q}(t)(\sqrt{q_1(t)}, \sqrt{q_2(t)})$$

is biquadratic over $\mathbb{Q}(t)$. Its degree-four map to $\mathbb{P}_t^1$ is branched at four distinct geometric points. Riemann–Hurwitz gives $2g(C) - 2 = 4(-2) + 8 = 0$, hence $g(C) = 1$.

At $t = t_0$ both radicands are squares:

$$\sqrt{q_1(t_0)} = \frac{6185869062}{11203}, \quad \sqrt{q_2(t_0)} = \frac{31426768294}{33609}. \quad (5)$$

This gives a rational point on C . To work with the parameter curve explicitly, parametrize the first conic by

$$t = \frac{N(u)}{D(u)}, \quad r_1 = \frac{W(u)}{D(u)}, \quad (6)$$

where

$$\begin{aligned} N(u) &= \frac{14974938550}{289} + \frac{550744148}{17}u + 4995486u^2, \\ D(u) &= \frac{1957773675}{17} + 74456844u + 11774081u^2, \\ W(u) &= \frac{18377134456116150}{289} + \frac{591506630529100}{17}u + 4746709944790u^2. \end{aligned}$$

The identity $q_1(N/D) = (W/D)^2$ holds coefficient by coefficient. The other equation becomes

$$w^2 = F(u), \quad F(u) = D(u)^2 q_2\left(\frac{N(u)}{D(u)}\right). \quad (7)$$

This squarefree quartic has a square constant term. Its five coefficients are in `data/pair.json` [3]. Equation (7), not either individual cover, is the genus-one quartic in the construction.

Write $F(u) = au^4 + bu^3 + cu^2 + du + e^2$, with $e > 0$. A Weierstrass model for C is

$$E_0 : Y^2 = X^3 + cX^2 + (bd - 4ae^2)X + e^2b^2 + ad^2 - 4ace^2. \quad (8)$$

The birational map and the coordinates of the rational point $P \in E_0(\mathbb{Q})$ corresponding to $u = 0$ are in the certificate [3]. In the next section, P is shown to have infinite order.

3 Independence of the nineteen sections

The exact section calculation is short to describe even though the coordinates are long. Let χ denote the holomorphic Euler characteristic of the elliptic surface. For a nonzero section Q on one of the models used here, write its x -coordinate in lowest terms as $x(Q) = N_Q/D_Q$. With only I_1 fibers, Shioda's height formula [5] takes the particularly simple form

$$h(Q) = \max\{\deg N_Q, \deg D_Q + 2\chi\}, \quad \langle Q, R \rangle = \frac{h(Q+R) - h(Q) - h(R)}{2}. \quad (9)$$

The parent has $\chi = 2$; either degree-two pullback has $\chi = 4$. The certificate checks the Weierstrass identities and the squarefree discriminant of degree 12χ before using (9). All of these calculations take place in exact rational-function arithmetic.

For the parent, the seventeen sections $P_1, \dots, P_{17}$ have a positive definite height matrix G with

$$\det G = 1152. \quad (10)$$

The first cover pulls them back and gives one additional section S_1 ; the second gives S_2 . The section coordinates are listed in `data/cover-4339.json` and `data/cover-36773.json` [3]. The labels in those file names are search identifiers, not degrees of the covers.

Model	Sections checked	Discriminant	Height determinant
Parent $E/\mathbb{Q}(t)$	17	squarefree, degree 24	1152
First quadratic pullback	18	squarefree, degree 48	452984832
Second quadratic pullback	18	squarefree, degree 48	452984832

Each 18×18 matrix is positive definite, and its upper-left 17×17 block is $2G$. The factor of two is the degree of the base change. This already proves rank at least 18 on each separate cover. To get nineteen on the compositum, one must also show that the two extra sections do not become dependent on each other.

Let σ_i change the sign of $\sqrt{q_i}$ and fix the other square root. Direct substitution into the recorded rational functions shows

$$h(S_1 - \sigma_1(S_1)) = 12, \quad h(S_2 - \sigma_2(S_2)) = 12. \quad (11)$$

Every inherited P_j is fixed by both involutions, while σ_1 fixes S_2 and σ_2 fixes S_1 . The two differences in (11) are nontorsion. Now suppose

$$n_1 S_1 + n_2 S_2 + \sum_{j=1}^{17} m_j P_j = O.$$

Applying $\sigma_1 - 1$ gives $n_1(\sigma_1(S_1) - S_1) = O$, so $n_1 = 0$. Applying $\sigma_2 - 1$ gives $n_2 = 0$. Positive definiteness of G then forces every $m_j = 0$. This proves the section assertion in Theorem 1.

4 Infinitely many trivial-torsion fibers

The rational point of C from (5) maps to a point $P \in E_0(\mathbb{Q})$. Its exact Weierstrass coordinates are stored as `auxiliary_point` in `data/pair.json` [3]. The curve E_0 has good reduction at 31 and 37, and exact point counts give

$$\#E_0(\mathbb{F}_{31}) = 30, \quad \#E_0(\mathbb{F}_{37}) = 32.$$

Rational torsion has order dividing two. The group law on (8) gives $2P \neq O$ exactly, so P has infinite order. Thus $C(\mathbb{Q})$ is infinite.

The family (2) is nonconstant. By the specialization theorem [6], its nineteen independent sections remain independent on all but finitely many fibers arising from the multiples of P . Since the map $C \rightarrow \mathbb{P}_t^1$ is nonconstant, those multiples yield infinitely many distinct rational values of t .

One can keep the torsion subgroup trivial along an infinite subsequence. At $t_0 = 15122/33609$, the fiber is smooth at the following primes:

p	31	37	59
$\#E_{t_0}(\mathbb{F}_p)$	42	46	71
$\#E_0(\mathbb{F}_p)$	30	32	72

The first row of orders has gcd one, hence $E_{t_0}(\mathbb{Q})$ has trivial torsion. The least common multiple of the second row is 1440. For every integer $n \equiv 1 \pmod{1440}$, the reductions of nP and P agree at all three primes. The map from E_0 to the t -line is regular at these reductions, so $E_{t(nP)}$ has the same three reduced fibers as E_{t_0} . Its rational torsion is therefore trivial. There are infinitely many such nP , and specialization retains the nineteen sections for all but finitely many of them. This finishes the proof of Theorem 1.

5 Proof data and verification

The proof data and verifier are available at a fixed revision of the *Rank Ledger* repository [3]. Four JSON files contain the parent model, the two pullbacks, and the paired-cover data. They include every coefficient and section coordinate used here. From the rank-19-genus-one-base-change directory, the verification command is

```
sage -python verify.py
```

It checks the section identities, the three height matrices, the cover maps and involutions, the auxiliary point, and the finite-field orders. The exact rank of the generic family, as opposed to the lower bound of nineteen, is not determined here.

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Use of language models

Large language models were used for literature review, computational cross-checking, and manuscript editing. The research direction, structural ideas, and computational implementation were developed by the author.

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